

# On canonical metrics on Cartan-Hartogs domains

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**Abstract** The Cartan-Hartogs domains are defined as a class of Hartogs type domains over irreducible bounded symmetric domains. The purpose of this paper is twofold. Firstly, for a Cartan-Hartogs domain  $\Omega^{B^{d_0}}(\mu)$  endowed with the canonical metric  $g(\mu)$ , we obtain an explicit formula for the Bergman kernel of the weighted Hilbert space  $\mathcal{H}_\alpha$  of square integrable holomorphic functions on  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  with the weight  $\exp\{-\alpha\varphi\}$  (where  $\varphi$  is a globally defined Kähler potential for  $g(\mu)$ ) for  $\alpha > 0$ , and, furthermore, we give an explicit expression of the Rawnsley's  $\varepsilon$ -function expansion for  $(\Omega^{B^{d_0}}(\mu), g(\mu))$ . Secondly, using the explicit expression of the Rawnsley's  $\varepsilon$ -function expansion, we show that the coefficient  $a_2$  of the Rawnsley's  $\varepsilon$ -function expansion for the Cartan-Hartogs domain  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is constant on  $\Omega^{B^{d_0}}(\mu)$  if and only if  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is biholomorphically isometric to the complex hyperbolic space. So we give an affirmative answer to a conjecture raised by M. Zedda.

**Key words:** Bounded symmetric domains · Cartan-Hartogs domains · Bergman kernels · Kähler metrics

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## 1 Introduction

The expansion of the Bergman kernel has received a lot of attention recently, due to the influential work of Donaldson, see e.g. [4], about the existence and uniqueness of constant scalar curvature Kähler metrics (cscK metrics). Donaldson used the asymptotics of the Bergman kernel proved by Catlin [3] and Zelditch [27] and the calculation of Lu [16] of the first coefficient in the expansion to give conditions for the existence of cscK metrics. This work inspired many papers on the subject since then. For the reference of the expansion of the Bergman kernel, see also Engliš [6], Loi [14], Ma-Marinescu [17, 18, 19], Xu [22] and references therein.

Assume that  $D$  is a bounded domain in  $\mathbb{C}^n$  and  $\varphi$  is a strictly plurisubharmonic function on  $D$ . Let  $g$  be a Kähler metric on  $D$  associated to the Kähler form  $\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi$ . For  $\alpha > 0$ , let  $\mathcal{H}_\alpha$  be the weighted Hilbert space of square integrable holomorphic functions on  $(D, g)$  with the weight  $\exp\{-\alpha\varphi\}$ , that is,

$$\mathcal{H}_\alpha := \left\{ f \in \text{Hol}(D) \mid \int_D |f|^2 \exp\{-\alpha\varphi\} \frac{\omega^n}{n!} < +\infty \right\},$$

where  $\text{Hol}(D)$  denotes the space of holomorphic functions on  $D$ . Let  $K_\alpha$  be the Bergman kernel (namely, the reproducing kernel) of  $\mathcal{H}_\alpha$  if  $\mathcal{H}_\alpha \neq \{0\}$ . The Rawnsley's  $\varepsilon$ -function on  $D$  (see Cahen-Gutt-Rawnsley [2] and Rawnsley [20]) associated to the metric  $g$  is defined by

$$\varepsilon_\alpha(z) := \exp\{-\alpha\varphi(z)\} K_\alpha(z, \bar{z}), \quad z \in D. \quad (1.1)$$

Note the Rawnsley's  $\varepsilon$ -function depends only on the metric  $g$  and not on the choice of the Kähler potential  $\varphi$  (which is defined up to an addition with the real part of a holomorphic function on  $D$ ). If the function  $\varepsilon_\alpha(z)$  ( $z \in D$ ) is a positive constant for  $\alpha = 1$ , the metric  $g$  on  $D$  is called to be balanced.

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The asymptotics of  $\varepsilon_\alpha$  was expressed in terms of the parameter  $\alpha$  for compact manifolds by Catlin [3] and Zelditch [27] (for  $\alpha \in \mathbb{N}$ ) and for non-compact manifolds by Ma-Marinescu [17, 18]. In some particular case it was also proved by Engliš [5, 6].

The Cartan-Hartogs domains are defined as a class of Hartogs type domains over irreducible bounded symmetric domains. Let  $\Omega$  be an irreducible bounded symmetric domain in  $\mathbb{C}^d$  of genus  $p$ . The generic norm of  $\Omega$  is defined by  $N(z, \bar{z}) := (V(\Omega)K(z, \bar{z}))^{-1/p}$ , where  $V(\Omega)$  is the total volume of  $\Omega$  with respect to the Euclidean measure of  $\mathbb{C}^d$  and  $K(z, \bar{z})$  is its Bergman kernel. For an irreducible bounded symmetric domain  $\Omega$  in  $\mathbb{C}^d$ , a positive real number  $\mu$  and a positive integer number  $d_0$ , the Cartan-Hartogs domain  $\Omega^{B^{d_0}}(\mu)$  is defined by

$$\Omega^{B^{d_0}}(\mu) := \left\{ (z, w) \in \Omega \times \mathbb{C}^{d_0} \subset \mathbb{C}^d \times \mathbb{C}^{d_0} \mid \|w\|^2 < N(z, \bar{z})^\mu \right\}, \quad (1.2)$$

where  $\|\cdot\|$  is the standard Hermitian norm in  $\mathbb{C}^{d_0}$ .

Let  $\mathcal{M}_{m,n}$  be the set of all  $m \times n$  matrices  $z = (z_{ij})$  with complex entries. Let  $\bar{z}$  be the complex conjugate of the matrix  $z$  and let  $z^t$  be the transpose of the matrix  $z$ .  $I$  denotes the identity matrix. If a square matrix  $z$  is positive definite, then we write  $z > 0$ . For each bounded classical symmetric domain  $\Omega$  (refer to Hua [12]), we list the genus  $p(\Omega)$ , the generic norm  $N_\Omega(z, \bar{z})$  of  $\Omega$  and corresponding Cartan-Hartogs domain  $\Omega^{B^{d_0}}(\mu)$  [23] according to its type as following.

(i) If  $\Omega = \Omega_I(m, n) := \{z \in \mathcal{M}_{m,n} : I - z\bar{z}^t > 0\}$  ( $1 \leq m \leq n$ ) (the classical domains of type I), then  $p(\Omega) = m + n$ ,  $N_\Omega(z, \bar{z}) = \det(I - z\bar{z}^t)$ , and

$$\Omega^{B^{d_0}}(\mu) = \left\{ (z, w) \in \Omega_I(m, n) \times \mathbb{C}^{d_0} \subset \mathbb{C}^{mn} \times \mathbb{C}^{d_0} : \|w\|^2 < (\det(I - z\bar{z}^t))^\mu \right\}.$$

Specially, when  $\Omega := B^n$  is the unit ball in  $\mathbb{C}^n$ , then we have

$$\Omega^{B^{d_0}}(\mu) = \left\{ (z, w) \in \Omega \times \mathbb{C}^{d_0} : \|z\|^2 + \|w\|^{\frac{2}{\mu}} < 1 \right\}.$$

It is a natural generalization of Thullen domains.

(ii) If  $\Omega = \Omega_{II}(n) := \{z \in \mathcal{M}_{n,n} : z^t = -z, I - z\bar{z}^t > 0\}$  ( $n \geq 4$ ) (the classical domains of type II), then  $p(\Omega) = 2(n - 1)$ ,  $N_\Omega(z, \bar{z}) = (\det(I - z\bar{z}^t))^{1/2}$ , and

$$\Omega^{B^{d_0}}(\mu) = \left\{ (z, w) \in \Omega_{II}(n) \times \mathbb{C}^{d_0} \subset \mathbb{C}^{n(n-1)/2} \times \mathbb{C}^{d_0} : \|w\|^2 < (\det(I - z\bar{z}^t))^\mu \right\}.$$

(iii) If  $\Omega = \Omega_{III}(n) := \{z \in \mathcal{M}_{n,n} : z^t = z, I - z\bar{z}^t > 0\}$  ( $n \geq 2$ ) (the classical domains of type III), then  $p(\Omega) = n + 1$ ,  $N_\Omega(z, \bar{z}) = \det(I - z\bar{z}^t)$ , and

$$\Omega^{B^{d_0}}(\mu) = \left\{ (z, w) \in \Omega_{III}(n) \times \mathbb{C}^{d_0} \subset \mathbb{C}^{n(n+1)/2} \times \mathbb{C}^{d_0} : \|w\|^2 < (\det(I - z\bar{z}^t))^\mu \right\}.$$

(iv) If  $\Omega = \Omega_{IV}(n) := \{z \in \mathbb{C}^n : 1 - 2z\bar{z}^t + |zz^t|^2 > 0, z\bar{z}^t < 1\}$  ( $n \geq 5$ ) (the classical domains of type IV), then  $p(\Omega) = n$ ,  $N_\Omega(z, \bar{z}) = 1 - 2z\bar{z}^t + |zz^t|^2$ , and

$$\Omega^{B^{d_0}}(\mu) = \left\{ (z, w) \in \Omega_{IV}(n) \times \mathbb{C}^{d_0} \subset \mathbb{C}^n \times \mathbb{C}^{d_0} : \|w\|^2 < (1 - 2z\bar{z}^t + |zz^t|^2)^\mu \right\}.$$

For the Cartan-Hartogs domain  $\Omega^{B^{d_0}}(\mu)$ , define

$$\Phi(z, w) := -\log(N(z, \bar{z})^\mu - \|w\|^2). \quad (1.3)$$

The Kähler form  $\omega(\mu)$  on  $\Omega^{B^{d_0}}(\mu)$  is defined by

$$\omega(\mu) := \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\Phi. \quad (1.4)$$

The Kähler metric  $g(\mu)$  on  $\Omega^{B^{d_0}}(\mu)$  associated to  $\omega(\mu)$  is given by  $ds^2 = \sum_{i,j=1}^n \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j$ , where  $n = d + d_0$ ,  $z = (z_1, z_2, \dots, z_d)$ ,  $w = (z_{d+1}, z_{d+2}, \dots, z_n)$ . With the exception of the complex hyperbolic space which is obviously homogeneous, each Cartan-Hartogs domain  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is a noncompact, nonhomogeneous, complete Kähler manifold (see Yin-Wang [25]). Further, for some particular value  $\mu_0$  of  $\mu$ ,  $g(\mu_0)$  is a Kähler-Einstein metric. For the general reference of the Cartan-Hartogs domains in this paper, see Loi-Zedda [15], Wang-Yin-Zhang-Roos [21], Yin [23], Yin-Wang [25], Zedda [26] and references therein.

In this paper, we study the asymptotics of the Rawnsley's  $\varepsilon$ -function on the Cartan-Hartogs domain with the canonical metric and draw some geometric consequences. For a Cartan-Hartogs domain  $(\Omega^{B^{d_0}}(\mu), g(\mu))$ , we have (see Theorem 3.1 in this paper) that the Rawnsley's  $\varepsilon$ -function admits the expansion:

$$\varepsilon_\alpha(z, w) = \sum_{j=0}^{d+d_0} a_j(z, w) \alpha^{d+d_0-j}, \quad (z, w) \in \Omega^{B^{d_0}}(\mu). \quad (1.5)$$

By Th. 1.1 of Lu [16], Th. 4.1.2 and Th. 6.1.1 of Ma-Marinescu [17], Th. 3.11 of Ma-Marinescu [18] and Th. 0.1 of Ma-Marinescu [19], see also Th. 3.3 of Xu [22], we have

$$\begin{cases} a_0 &= 1, \\ a_1 &= \frac{1}{2}k_g, \\ a_2 &= \frac{1}{3}\Delta k_g + \frac{1}{24}|R|^2 - \frac{1}{6}|Ric|^2 + \frac{1}{8}k_g^2, \end{cases} \quad (1.6)$$

where  $k_g$ ,  $\Delta$ ,  $R$  and  $Ric$  denote the scalar curvature, the Laplace, the curvature tensor and the Ricci curvature associated to the metric  $g(\mu)$ , respectively.

Let  $B^d := \{z = (z_1, z_2, \dots, z_d) \in \mathbb{C}^d \mid \|z\|^2 = \sum_{k=1}^d |z_k|^2 < 1\}$  and let the metric  $g_{hyp}$  on  $B^d$  be given by  $ds^2 = -\sum_{i,j=1}^d \frac{\partial^2 \log(1-\|z\|^2)}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j$ . Then we call  $(B^d, g_{hyp})$  the complex hyperbolic space. Note that here  $\mathcal{H}_\alpha \neq \{0\}$  iff  $\alpha > d$  and that  $\alpha g_{hyp}$  ( $\alpha > 0$ ) is a balanced metric on  $B^d$  iff  $\alpha > d$ .

Loi and Zedda [15] studied balanced metrics on the Cartan-Hartogs domain and proved the following result for  $d_0 = 1$ :

**Theorem 1.1.** (Loi-Zedda [15] for  $d_0 = 1$ ) *Let  $\Omega$  be an irreducible bounded symmetric domain of dimension  $d$  and genus  $p$ . Then the metric  $\alpha g(\mu)$  on  $\Omega^{B^{d_0}}(\mu)$  is balanced if and only if  $\alpha > \max\{d + d_0, \frac{p-1}{\mu}\}$  and  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is holomorphically isometric to the complex hyperbolic space  $(B^{d+d_0}, g_{hyp})$ , namely,  $\Omega = B^d$  and  $\mu = 1$ .*

By calculating the scalar curvature  $k_g$ , the Laplace  $\Delta k_g$  of  $k_g$ , the norm  $|R|^2$  of the curvature tensor  $R$  and the norm  $|Ric|^2$  of the Ricci curvature  $Ric$  of a Cartan-Hartogs domain  $(\Omega^{B^{d_0}}(\mu), g(\mu))$ , Zedda [26] has proved the following theorem for  $d_0 = 1$ :

**Theorem 1.2.** (Zedda [26] for  $d_0 = 1$ ) *Let  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  be a Cartan-Hartogs domain. If the coefficient  $a_2$  of the Rawnsley's  $\varepsilon$ -function expansion is a constant on  $\Omega^{B^{d_0}}(\mu)$ , then  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is Kähler-Einstein.*

Further, Zedda [26] conjectured that the coefficient  $a_2$  of the expansion of the Rawnsley's  $\varepsilon$ -function associated to  $g(\mu)$  is constant if and only if  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is biholomorphically isometric to the complex hyperbolic space. Obviously, the conjecture implies Theorem 1.2. In this paper, for any positive integer  $d_0$ , by giving an explicit expression of the reproducing kernel  $K_\alpha$  of  $\mathcal{H}_\alpha$  and the Rawnsley's  $\varepsilon$ -function for  $(\Omega^{B^{d_0}}(\mu), g(\mu))$ , we prove that the Zedda's conjecture is affirmative, namely, we prove the following conclusion:

**Theorem 1.3.** *Let  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  be a Cartan-Hartogs domain. Then the coefficient  $a_2$  of the Rawnsley's  $\varepsilon$ -function expansion is a constant on  $\Omega^{B^{d_0}}(\mu)$  if and only if  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is biholomorphically isometric to the complex hyperbolic space  $(B^{d+d_0}, g_{hyp})$ .*

Remark that Theorem 1.3 immediately implies Theorem 1.1. In fact, for  $\alpha > \max\{d + d_0, \frac{p-1}{\mu}\}$ , let  $\omega_\alpha := \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(-\alpha \log(N(z, \bar{z})^\mu - \|w\|^2)) = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(\alpha\Phi)$  (so  $\omega_\alpha = \alpha\omega(\mu)$ ),

$$\mathcal{H}_{\omega_\alpha} := \left\{ f \in \text{Hol}(\Omega^{B^{d_0}}(\mu)) \mid \int_{\Omega^{B^{d_0}}(\mu)} |f|^2 \exp\{-\alpha\Phi\} \frac{\omega_\alpha^{d+d_0}}{(d+d_0)!} < +\infty \right\},$$

and

$$\mathcal{H}_\alpha := \left\{ f \in \text{Hol}(\Omega^{B^{d_0}}(\mu)) \mid \int_{\Omega^{B^{d_0}}(\mu)} |f|^2 \exp\{-\alpha\Phi\} \frac{\omega(\mu)^{d+d_0}}{(d+d_0)!} < +\infty \right\}.$$

It is easy to see that  $K_{\omega_\alpha} = \frac{1}{\alpha^{d+d_0}} K_\alpha$ , where  $K_{\omega_\alpha}$  and  $K_\alpha$  are the Bergman kernels of  $\mathcal{H}_{\omega_\alpha}$  and  $\mathcal{H}_\alpha$ , respectively. So, we have  $\exp\{-\alpha\Phi\} K_{\omega_\alpha} = \frac{1}{\alpha^{d+d_0}} \exp\{-\alpha\Phi\} K_\alpha$ . By the definition,  $\alpha g(\mu)$  on  $\Omega^{B^{d_0}}(\mu)$  is balanced iff  $\exp\{-\alpha\Phi\} K_{\omega_\alpha}$  is a positive constant on  $\Omega^{B^{d_0}}(\mu)$ . This indicate that  $\alpha g(\mu)$  on  $\Omega^{B^{d_0}}(\mu)$  is balanced if and only if  $\varepsilon_\alpha = \exp\{-\alpha\Phi\} K_\alpha$  is a positive constant on  $\Omega^{B^{d_0}}(\mu)$ . Note that  $\alpha g_{hyp}$  is a balanced metric on  $B^d$  iff  $\alpha > d$ , and if the metric  $\alpha g(\mu)$  on  $\Omega^{B^{d_0}}(\mu)$  is balanced, then we have  $\alpha > \max\{d + d_0, \frac{p-1}{\mu}\}$  (see Lemma 9 in [15]). Thus, by Theorem 1.3, we obtain that the metric  $\alpha g(\mu)$  on  $\Omega^{B^{d_0}}(\mu)$  is balanced if and only if  $\alpha > \max\{d + d_0, \frac{p-1}{\mu}\}$  and  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is holomorphically isometric to the complex hyperbolic space  $(B^{d+d_0}, g_{hyp})$ . The proof is complete.

On the other hand, combining the formulas (3.2) and (3.25) in this paper, we have that  $a_1$  is constant if and only if  $\mu = \frac{p}{d+1}$ . Further, by (1.6), the scalar curvature of a Cartan-Hartogs domain  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is constant iff  $a_1$  is constant. Thus we get the following theorem:

**Theorem 1.4.** (Zedda [26] for  $d_0 = 1$ ) *Let  $\Omega$  be an irreducible bounded symmetric domain of dimension  $d$  and genus  $p$ . Then the scalar curvature of a Cartan-Hartogs domain  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is constant if and only if  $g(\mu)$  is Kähler-Einstein, namely,  $\mu = \frac{p}{d+1}$ .*

The paper is organized as follows. In Section 2, we obtain an explicit formula for the Bergman kernel  $K_\alpha$  of  $\mathcal{H}_\alpha$  for the Cartan-Hartogs domain  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  in terms of ranks, Hua polynomials and generic norms of  $\Omega$  and  $B^{d_0}$  (Theorem 2.3). In Section 3, using results in Section 2, we give the explicit expansion of the Rawnsley's  $\varepsilon$ -function and obtain the expression of its coefficients  $a_1, a_2$  for the Cartan-Hartogs domain associated to  $g(\mu)$  (Corollary 3.2). Finally, in Section 4, the conclusion is achieved by using the classification of bounded symmetric domains (it follows that  $a_2$  is constant if and only if the rank  $r = 1$  and  $\mu = 1$ ).

## 2 The reproducing kernel of $\mathcal{H}_\alpha$ for $\Omega^{B^{d_0}}(\mu)$ with the canonical metric $g(\mu)$

Let  $\Omega$  be an irreducible bounded symmetric domain in  $\mathbb{C}^d$  in its Harish-Chandra realization. Thus  $\Omega$  is the open unit ball of a Banach space which admits the structure of a  $JB^*$ -triple. We denote  $r, a, b, d, p$  and  $N(z, \bar{w})$  by the rank, the characteristic multiplicities, the dimension, the genus, and the generic norm of  $\Omega$ , respectively. Thus

$$d = \frac{r(r-1)}{2}a + rb + r, \quad p = (r-1)a + b + 2. \quad (2.1)$$

For any  $s > -1$ , the value of the Hua integral  $\int_\Omega N(z, \bar{z})^s dm(z)$  is given by

$$\int_\Omega N(z, \bar{z})^s dm(z) = \frac{\chi(0)}{\chi(s)} \int_\Omega dm(z), \quad (2.2)$$

where  $dm(z)$  denotes the Euclidean measure on  $\mathbb{C}^d$ ,  $\chi$  is the Hua polynomial

$$\chi(s) := \prod_{j=1}^r \left( s + 1 + (j-1)\frac{a}{2} \right)_{1+b+(r-j)a}, \quad (2.3)$$

in which, for a non-negative integer  $m$ ,  $(s)_m$  denotes the raising factorial

$$(s)_m := \frac{\Gamma(s+m)}{\Gamma(s)} = s(s+1) \cdots (s+m-1).$$

Let  $\mathcal{G}$  stand for the identity connected component of the group of biholomorphic self-maps of  $\Omega$ , and  $\mathcal{K}$  for the stabilizer of the origin in  $\mathcal{G}$ . Under the action  $f \mapsto f \circ k$  ( $k \in \mathcal{K}$ ) of  $\mathcal{K}$ , the space  $\mathcal{P}$  of holomorphic polynomials on  $\mathbb{C}^d$  admits the Peter-Weyl decomposition

$$\mathcal{P} = \bigoplus_{\lambda} \mathcal{P}_{\lambda},$$

where the summation is taken over all partitions  $\lambda$ , i.e.,  $r$ -tuples  $(\lambda_1, \lambda_2, \dots, \lambda_r)$  of nonnegative integers such that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$ , and the spaces  $\mathcal{P}_{\lambda}$  are  $\mathcal{K}$ -invariant and irreducible. For each  $\lambda$ ,  $\mathcal{P}_{\lambda} \subset \mathcal{P}_{|\lambda|}$ , where  $|\lambda|$  denotes the weight of partition  $\lambda$ , i.e.,  $|\lambda| := \sum_{j=1}^r \lambda_j$ , and  $\mathcal{P}_{|\lambda|}$  is the space of homogeneous holomorphic polynomials of degree  $|\lambda|$ .

Let

$$\langle f, g \rangle_{\mathcal{F}} := \int_{\mathbb{C}^d} f(z) \overline{g(z)} d\rho_{\mathcal{F}}(z) \quad (2.4)$$

be the Fock-Fischer inner product on the space  $\mathcal{P}$  of holomorphic polynomials on  $\mathbb{C}^d$ , where

$$d\rho_{\mathcal{F}}(z) := \frac{1}{\pi^d} e^{-\|z\|^2} dm(z). \quad (2.5)$$

For every partition  $\lambda$ , let  $K_{\lambda}(z_1, \overline{z_2})$  be the Bergman kernel of  $\mathcal{P}_{\lambda}$  with respect to (2.4). The weighted Bergman kernel of the weighted Hilbert space  $A^2(\mathbb{C}^d, \rho_{\mathcal{F}})$  of square-integrable holomorphic functions on  $\mathbb{C}^d$  with the measure  $d\rho_{\mathcal{F}}$  is

$$K(z_1, \overline{z_2}) := \sum_{\lambda} K_{\lambda}(z_1, \overline{z_2}). \quad (2.6)$$

The kernels  $K_{\lambda}(z_1, \overline{z_2})$  are related to the generic norm  $N(z_1, \overline{z_2})$  by the Faraut-Korányi formula

$$N(z_1, \overline{z_2})^{-s} = \sum_{\lambda} (s)_{\lambda} K_{\lambda}(z_1, \overline{z_2}), \quad (2.7)$$

the series converges uniformly on compact subsets of  $\Omega \times \Omega$ ,  $s \in \mathbb{C}$ , where  $(s)_{\lambda}$  denote the generalized Pochhammer symbol

$$(s)_{\lambda} := \prod_{j=1}^r \left( s - \frac{j-1}{2}a \right)_{\lambda_j}. \quad (2.8)$$

For the proofs of above facts and additional details, we refer, e.g., to [7], [8] and [24].

**Lemma 2.1.** *Let  $\Omega$  be an irreducible bounded symmetric domain in  $\mathbb{C}^d$  in its Harish-Chandra realization with the generic norm  $N$  and the genus  $p$ . For  $z_0 \in \Omega$ , let  $\phi$  be an automorphism of  $\Omega$  such that  $\phi(z_0) = 0$ . By [21], the function*

$$\psi(z) := \frac{N(z_0, \overline{z_0})^{\frac{\mu}{2}}}{N(z, \overline{z_0})^{\mu}} \quad (2.9)$$

satisfies

$$|\psi(z)|^2 = \left( \frac{N(\phi(z), \overline{\phi(z)})}{N(z, \bar{z})} \right)^\mu. \quad (2.10)$$

Define the mapping  $F$

$$\begin{aligned} F : \Omega^{B^{d_0}}(\mu) &\longrightarrow \Omega^{B^{d_0}}(\mu), \\ (z, w) &\longmapsto (\phi(z), \psi(z)w). \end{aligned} \quad (2.11)$$

Then  $F$  is an isometric automorphism of  $(\Omega^{B^{d_0}}(\mu), g(\mu))$ , that is

$$\partial\bar{\partial}(\Phi(F(z, w))) = \partial\bar{\partial}(\Phi(z, w)). \quad (2.12)$$

*Proof.* From [21], we know that  $F$  is an automorphism of  $\Omega^{B^{d_0}}(\mu)$ , and

$$N(\phi(z), \overline{\phi(z)})^p = J\phi(z)N(z, \bar{z})^p \overline{J\phi(z)}, \quad (2.13)$$

where  $J\phi(z)$  is the Jacobian of  $\phi$ .

By (2.10) and (2.13), we have

$$\begin{aligned} &N(\phi(z), \overline{\phi(z)})^\mu - \|\psi(z)w\|^2 \\ &= N(\phi(z), \overline{\phi(z)})^\mu \left( 1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu} \right) \\ &= |J\phi(z)|^{\frac{2\mu}{p}} (N(z, \bar{z})^\mu - \|w\|^2), \end{aligned}$$

which implies (2.12).  $\square$

**Lemma 2.2.** Let  $\Omega$  be the Cartan domain with the generic norm  $N(z, \bar{\xi})$ , the dimension  $d$  and the genus  $p$ . Then we have

$$\det \left( \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^n (z, w) = \frac{\mu^d C_\Omega N(z, \bar{z})^{\mu(d+1)-p}}{(N(z, \bar{z})^\mu - \|w\|^2)^{n+1}}, \quad (2.14)$$

where the function  $\Phi(z, w) = -\log(N(z, \bar{z})^\mu - \|w\|^2)$ ,  $n = d + d_0$ ,  $z = (z_1, z_2, \dots, z_d)$ ,  $w = (w_1, w_2, \dots, w_{d_0}) = (z_{d+1}, z_{d+2}, \dots, z_n)$ , and  $C_\Omega = \det \left( -\frac{\partial^2 \log N(z, \bar{z})}{\partial z_i \partial \bar{z}_j} \right) \Big|_{z=0}$ .

*Proof.* It is well known that

$$\frac{(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\Phi)^n}{n!} = \det \left( \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^n \frac{\omega_0^n}{n!}, \quad (2.15)$$

where  $\omega_0 = \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ .

From (2.12) and (2.15), we get

$$\det \left( \frac{\partial^2 \Phi(F)}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^n = \det \left( \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^n. \quad (2.16)$$

By the identity

$$\left( \frac{\partial^2 \Phi(F)}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^n (z, w) = \left( \frac{\partial F_j}{\partial z_i} \right)_{i,j=1}^n \left( \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^n (F(z, w)) \left( \frac{\partial \bar{F}_i}{\partial \bar{z}_j} \right)_{i,j=1}^n \quad (2.17)$$

and (2.16), we deduce

$$\det \left( \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^n (z, w) = |JF(z, w)|^2 \det \left( \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^n (F(z, w)), \quad (2.18)$$

where

$$JF(z, w) := \det \left( \frac{\partial F_j}{\partial z_i} \right)_{i,j=1}^n,$$

and

$$\left( \frac{\partial F_j}{\partial z_i} \right)_{i,j=1}^n := \begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \frac{\partial F_2}{\partial z_1} & \cdots & \frac{\partial F_n}{\partial z_1} \\ \frac{\partial F_1}{\partial z_2} & \frac{\partial F_2}{\partial z_2} & \cdots & \frac{\partial F_n}{\partial z_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_1}{\partial z_n} & \frac{\partial F_2}{\partial z_n} & \cdots & \frac{\partial F_n}{\partial z_n} \end{pmatrix}.$$

Let  $(\widetilde{z}_0, \widetilde{w}_0) = F(z_0, w_0)$ ,  $(z_0, w_0) \in \Omega^{B^{d_0}}(\mu)$ . By (2.9) and (2.11), then  $(\widetilde{z}_0, \widetilde{w}_0) = \left(0, \frac{w_0}{N(z_0, \overline{z}_0)^{\frac{\mu}{2}}}\right)$  and

$$|JF(z_0, w_0)|^2 = |J\phi(z_0)|^2 |\psi(z_0)|^{2d_0}. \quad (2.19)$$

Using  $N(0, z) = 1$ , (2.9), (2.13), (2.19) and (2.18), we have

$$|JF(z_0, w_0)|^2 = \frac{1}{N(z_0, \overline{z}_0)^{p+\mu d_0}}, \quad (2.20)$$

and

$$\det \left( \frac{\partial^2 \Phi}{\partial z_i \partial \overline{z}_j} \right)_{i,j=1}^n (z_0, w_0) = \frac{1}{N(z_0, \overline{z}_0)^{p+\mu d_0}} \det \left( \frac{\partial^2 \Phi}{\partial z_i \partial \overline{z}_j} \right)_{i,j=1}^n (0, \widetilde{w}_0). \quad (2.21)$$

Now we calculate  $\det \left( \frac{\partial^2 \Phi}{\partial z_i \partial \overline{z}_j} \right)_{i,j=1}^n (0, w)$ .

From

$$\begin{cases} N(0, 0) &= 1, \\ \frac{\partial \log N(z, \overline{z})}{\partial z_i} \Big|_{z=0} &= \frac{\partial \log N(z, \overline{z})}{\partial \overline{z}_i} \Big|_{z=0} = 0 \quad (1 \leq i \leq d), \end{cases} \quad (2.22)$$

we obtain

$$\begin{cases} \frac{\partial N(z, \overline{z})^\mu}{\partial z_i} \Big|_{z=0} &= \frac{\partial}{\partial z_i} \exp\{\mu \log N(z, \overline{z})\} \Big|_{z=0} = \mu N(z, \overline{z})^\mu \frac{\partial \log N(z, \overline{z})}{\partial z_i} \Big|_{z=0} = 0, \\ \frac{\partial N(z, \overline{z})^\mu}{\partial \overline{z}_i} \Big|_{z=0} &= 0, \\ \frac{\partial^2 N(z, \overline{z})^\mu}{\partial z_i \partial \overline{z}_j} \Big|_{z=0} &= \left\{ \mu N(z, \overline{z})^\mu \frac{\partial^2 \log N(z, \overline{z})}{\partial z_i \partial \overline{z}_j} + \mu \frac{\partial \log N(z, \overline{z})}{\partial z_i} \frac{\partial N(z, \overline{z})^\mu}{\partial \overline{z}_j} \right\} \Big|_{z=0} = -\mu c_{ij}. \end{cases} \quad (2.23)$$

where  $c_{ij} = - \frac{\partial^2 \log N(z, \overline{z})}{\partial z_i \partial \overline{z}_j} \Big|_{z=0}$ . Therefore

$$\begin{cases} \frac{\partial^2 \Phi}{\partial z_i \partial \overline{z}_j}(0, w) &= \left\{ \frac{1}{(N(z, \overline{z})^\mu - \|w\|^2)^2} \frac{\partial N(z, \overline{z})^\mu}{\partial z_i} \frac{\partial N(z, \overline{z})^\mu}{\partial \overline{z}_j} - \frac{1}{N(z, \overline{z})^\mu - \|w\|^2} \frac{\partial^2 N(z, \overline{z})^\mu}{\partial z_i \partial \overline{z}_j} \right\} \Big|_{z=0} \\ &= \frac{\mu c_{ij}}{1 - \|w\|^2} \quad (1 \leq i, j \leq d), \\ \frac{\partial^2 \Phi}{\partial w_i \partial \overline{w}_j}(0, w) &= \left\{ \frac{\delta_{ij}}{N(z, \overline{z})^\mu - \|w\|^2} + \frac{\overline{w}_i w_j}{(N(z, \overline{z})^\mu - \|w\|^2)^2} \right\} \Big|_{z=0} \\ &= \frac{\delta_{ij}}{1 - \|w\|^2} + \frac{\overline{w}_i w_j}{(1 - \|w\|^2)^2} \quad (1 \leq i, j \leq d_0), \\ \frac{\partial^2 \Phi}{\partial z_i \partial \overline{w}_j}(0, w) &= \frac{-w_j}{(N(z, \overline{z})^\mu - \|w\|^2)^2} \frac{\partial N(z, \overline{z})^\mu}{\partial z_i} \Big|_{z=0} = 0 \quad (1 \leq i \leq d, 1 \leq j \leq d_0), \\ \frac{\partial^2 \Phi}{\partial w_i \partial \overline{z}_j}(0, w) &= \frac{-\overline{w}_i}{(N(z, \overline{z})^\mu - \|w\|^2)^2} \frac{\partial N(z, \overline{z})^\mu}{\partial \overline{z}_j} \Big|_{z=0} = 0 \quad (1 \leq i \leq d_0, 1 \leq j \leq d). \end{cases} \quad (2.24)$$

The above results can be rewritten as

$$\left( \frac{\partial^2 \Phi}{\partial z_i \partial \overline{z}_j} \right)_{i,j=1}^n (0, w) = \begin{pmatrix} \frac{\mu}{1 - \|w\|^2} C_d & 0 \\ 0 & \frac{1}{1 - \|w\|^2} I_{d_0} + \frac{1}{(1 - \|w\|^2)^2} w^\dagger w \end{pmatrix}, \quad (2.25)$$



where  $I_{d_0}$  denotes the  $d_0 \times d_0$  diagonal matrix with its diagonal elements 1,  $w^\dagger$  is the conjugate transpose of the row vector  $w = (w_1, w_2, \dots, w_{d_0})$ , and  $C_d = (c_{ij})_{i,j=1}^d$ .

From (2.25), we have

$$\det \left( \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^n (0, w) = \frac{\mu^d \det C_d}{(1 - \|w\|^2)^{d+d_0+1}}. \quad (2.26)$$

Finally, by (2.21) and (2.26), we have (2.14).  $\square$

**Theorem 2.3.** *Let  $\Omega^{B^{d_0}}(\mu)$  be a Cartan-Hartogs domain with the Kähler metric  $g(\mu)$  (see (1.4)). Let  $\alpha > \max\{d + d_0, \frac{p-1}{\mu}\}$ . Then the Bergman kernel  $K_\alpha(z, w; \bar{z}, \bar{w})$  of the Hilbert space*

$$\mathcal{H}_\alpha = \left\{ f \in \text{Hol}(\Omega^{B^{d_0}}(\mu)) \mid \int_{\Omega^{B^{d_0}}(\mu)} |f|^2 \exp\{-\alpha\Phi\} \frac{\omega(\mu)^{d+d_0}}{(d+d_0)!} < +\infty \right\}$$

can be written as

$$\begin{aligned} & K_\alpha(z, w; \bar{z}, \bar{w}) \\ &= \frac{\pi^{d+d_0} \chi_2(\alpha - d - d_0 - 1)}{C_\Omega \mu^d \chi_1(0) \chi_2(0) V(\Omega) V(B^{d_0})} \left( \frac{1}{N(z, \bar{z})} \right)^{\mu\alpha} \chi_1\left(\mu\left(\alpha + t \frac{d}{dt}\right) - p\right) \frac{1}{\left(1 - t \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right)^{\alpha-d}} \Bigg|_{t=1} \end{aligned} \quad (2.27)$$

where  $C_\Omega = \det\left(-\frac{\partial^2 \log N(z, \bar{z})}{\partial z_i \partial \bar{z}_j}\right)\Big|_{z=0}$ ,  $\chi_1, \chi_2$  and  $V(\Omega), V(B^{d_0})$  are Hua polynomials (see (2.3)) and the volumes with respect to the Euclidean measure of  $\Omega, B^{d_0}$ , respectively.

*Proof.* By (2.14), the inner product on  $\mathcal{H}_\alpha$  is given by

$$(f, g) = \frac{\mu^d C_\Omega}{\pi^{d+d_0}} \int_{\Omega^{B^{d_0}}(\mu)} f(z, w) \overline{g(z, w)} N(z, \bar{z})^{\mu(\alpha-d_0)-p} \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right)^{\alpha-(d+d_0+1)} dm(z) dm(w),$$

where  $dm$  denotes the Euclidean measure.

For convenience, we set  $\Omega_1 = \Omega, \Omega_2 = B^{d_0}$ . Let  $r_i, a_i, b_i, d_i, p_i, \chi_i, (s)_\lambda^{(i)}$  and  $N_i$  be rank, characteristic multiplicities, dimension, genus, Hua polynomial, generalized Pochhammer symbol and generic norm of the irreducible bounded symmetric domain  $\Omega_i$ ,  $1 \leq i \leq 2$ .

Let  $\mathcal{G}_i$  stand for the identity connected components of groups of biholomorphic self-maps of  $\Omega_i \subset \mathbb{C}^{d_i}$ , and  $\mathcal{K}_i$  for stabilizer of the origin in  $\mathcal{G}_i$ , respectively. For any  $k = (k_1, k_2) \in \mathcal{K} := \mathcal{K}_1 \times \mathcal{K}_2$ , we define the action

$$\pi(k)f(z, w) \equiv f \circ k(z, w) := f(k_1 \circ z, k_2 \circ w)$$

of  $\mathcal{K}$ , then the space  $\mathcal{P}$  of holomorphic polynomials on  $\mathbb{C}^{d_1} \times \mathbb{C}^{d_2}$  admits the Peter-Weyl decomposition

$$\mathcal{P} = \bigoplus_{\substack{\ell(\lambda) \leq r_1 \\ \ell(\nu) \leq r_2}} \mathcal{P}_\lambda^{(1)} \otimes \mathcal{P}_\nu^{(2)},$$

where spaces  $\mathcal{P}_\lambda^{(i)}$  are  $\mathcal{K}_i$ -invariant and irreducible subspaces of spaces of holomorphic polynomials on  $\mathbb{C}^{d_i}$  ( $1 \leq i \leq 2$ ).

Since  $\mathcal{H}_\alpha$  is invariant under the action of  $\mathcal{K}_1 \times \mathcal{K}_2$ , namely,  $\forall k \in \mathcal{K}_1 \times \mathcal{K}_2, (\pi(k)f, \pi(k)g) = (f, g)$ ,  $\mathcal{H}_\alpha$  admits an irreducible decomposition (see [9])

$$\mathcal{H}_\alpha = \widehat{\bigoplus_{\substack{\ell(\lambda) \leq r_1 \\ \ell(\nu) \leq r_2}} \mathcal{P}_\lambda^{(1)} \otimes \mathcal{P}_\nu^{(2)}},$$



where  $\widehat{\oplus}$  denotes the orthogonal direct sum.

For every partition  $\lambda$  of length  $\leq r_i$ , let  $K_\lambda^{(i)}(z, \bar{w})$  be the Bergman kernel of  $\mathcal{P}_\lambda^{(i)}$  with respect to (2.4). By Schur's lemma, there exist positive constants  $c_{\lambda\nu}$  such that  $c_{\lambda\nu}K_\lambda^{(1)}(z, \bar{z})K_\nu^{(2)}(w, \bar{w})$  are the reproducing kernels of  $\mathcal{P}_\lambda^{(1)} \otimes \mathcal{P}_\nu^{(2)}$  with respect to the above inner product  $(\cdot, \cdot)$ . According to the definition of the reproducing kernel, we have

$$\begin{aligned} & \frac{\mu^d C_\Omega}{\pi^{d+d_0}} \int_{\Omega^{B^{d_0}}(\mu)} c_{\lambda\nu} K_\lambda^{(1)}(z, \bar{z}) K_\nu^{(2)}(w, \bar{w}) N(z, \bar{z})^{\mu(\alpha-d_0)-p} \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right)^{\alpha-(d+d_0+1)} dm(z) dm(w) \\ &= \dim \mathcal{P}_\lambda^{(1)} \dim \mathcal{P}_\nu^{(2)}. \end{aligned}$$

Therefore, the Bergman kernel of  $\mathcal{H}_\alpha$  can be written as

$$K_\alpha(z, w; \bar{z}, \bar{w}) = \sum_{\substack{\ell(\lambda) \leq r_1 \\ \ell(\nu) \leq r_2}} \frac{\dim \mathcal{P}_\lambda^{(1)} \dim \mathcal{P}_\nu^{(2)}}{\langle K_\lambda^{(1)}(z, \bar{z}) K_\nu^{(2)}(w, \bar{w}) \rangle} K_\lambda^{(1)}(z, \bar{z}) K_\nu^{(2)}(w, \bar{w}), \quad (2.28)$$

where  $\langle f \rangle$  denotes integral

$$\frac{\mu^d C_\Omega}{\pi^{d+d_0}} \int_{\Omega^{B^{d_0}}(\mu)} f(z, w) N(z, \bar{z})^{\mu(\alpha-d_0)-p} \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right)^{\alpha-(d+d_0+1)} dm(z) dm(w).$$

If  $\mu\alpha - p > -1$  and  $\alpha - d - d_0 - 1 > -1$  (namely,  $\alpha > \max\{d + d_0, \frac{p-1}{\mu}\}$ ), combining (see [10], [11])

$$\int_{\Omega} K_\lambda(z, \bar{z}) N(z, \bar{z})^s dm(z) = \frac{\dim \mathcal{P}_\lambda}{(p+s)_\lambda} \int_{\Omega} N(z, \bar{z})^s dm(z) \quad (2.29)$$

for  $s > -1$  and (2.2), we have

$$\begin{aligned} & \frac{\mu^d C_\Omega}{\pi^{d+d_0}} \int_{\Omega^{B^{d_0}}(\mu)} K_\lambda^{(1)}(z, \bar{z}) K_\nu^{(2)}(w, \bar{w}) N(z, \bar{z})^{\mu(\alpha-d_0)-p} \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right)^{\alpha-(d+d_0+1)} dm(z) dm(w) \\ &= \frac{\mu^d C_\Omega}{\pi^{d+d_0}} \int_{\Omega} K_\lambda^{(1)}(z, \bar{z}) N(z, \bar{z})^{\mu(\alpha+\nu)-p} dm(z) \int_{B^{d_0}} K_\nu^{(2)}(w, \bar{w}) (1 - \|w\|^2)^{\alpha-(d+d_0+1)} dm(w) \\ &= \frac{\mu^d C_\Omega \chi_1(0) \chi_2(0) V(\Omega) V(B^{d_0})}{\pi^{d+d_0} \chi_1(\mu(\alpha+\nu)-p) \chi_2(\alpha-(d+d_0+1))} \frac{\dim \mathcal{P}_\lambda^{(1)} \dim \mathcal{P}_\nu^{(2)}}{(\mu(\alpha+\nu))_\lambda^{(1)} (\alpha-d)_\nu^{(2)}}. \end{aligned} \quad (2.30)$$

Combing (2.28), (2.30) and (2.7) we get

$$\begin{aligned}
& K_\alpha(z, w; \bar{z}, \bar{w}) \\
&= \sum_{\substack{\ell(\lambda) \leq r_1 \\ \ell(\nu) \leq r_2}} c \chi_1(\mu(\alpha + \nu) - p)(\mu(\alpha + \nu))_\lambda^{(1)} (\alpha - d)_\nu^{(2)} K_\lambda^{(1)}(z, \bar{z}) K_\nu^{(2)}(w, \bar{w}) \\
&= c \sum_{\ell(\nu) \leq r_2} \chi_1(\mu(\alpha + \nu) - p)(\alpha - d)_\nu^{(2)} K_\nu^{(2)}(w, \bar{w}) \frac{1}{N(z, \bar{z})^{\mu(\alpha + \nu)}} \\
&= \frac{c}{N(z, \bar{z})^{\mu\alpha}} \sum_{\ell(\nu) \leq r_2} \chi_1(\mu(\alpha + \nu) - p)(\alpha - d)_\nu^{(2)} K_\nu^{(2)}\left(\frac{w}{N(z, \bar{z})^\mu}, \bar{w}\right) \\
&= \frac{c}{N(z, \bar{z})^{\mu\alpha}} \sum_{\ell(\nu) \leq r_2} \chi_1\left(\mu\left(\alpha + t \frac{d}{dt}\right) - p\right)(\alpha - d)_\nu^{(2)} K_\nu^{(2)}\left(\frac{tw}{N(z, \bar{z})^\mu}, \bar{w}\right) \Bigg|_{t=1} \\
&= \frac{c}{N(z, \bar{z})^{\mu\alpha}} \chi_1\left(\mu\left(\alpha + t \frac{d}{dt}\right) - p\right) \sum_{\ell(\nu) \leq r_2} (\alpha - d)_\nu^{(2)} K_\nu^{(2)}\left(\frac{tw}{N(z, \bar{z})^\mu}, \bar{w}\right) \Bigg|_{t=1} \\
&= \frac{c}{N(z, \bar{z})^{\mu\alpha}} \chi_1\left(\mu\left(\alpha + t \frac{d}{dt}\right) - p\right) \frac{1}{\left(1 - \frac{t\|w\|^2}{N(z, \bar{z})^\mu}\right)^{\alpha-d}} \Bigg|_{t=1},
\end{aligned}$$

where

$$c = \frac{\pi^{d+d_0} \chi_2(\alpha - (d + d_0 + 1))}{\mu^d C_\Omega \chi_1(0) \chi_2(0) V(\Omega) V(B^{d_0})},$$

which completes the proof.  $\square$

In order to simplify (2.27), we need Lemma 2.4 below.

**Lemma 2.4.** (see [10]) *Let  $\varphi(x)$  be a polynomial in  $x$  of degree  $n$  and let  $Z$  be a matrix of order  $m$ . Let  $t$  be a real variable such that  $\|tZ\| < 1$ , where  $\|Z\|$  denotes the norm of  $Z$ . For a real number  $n_0$ , take  $x_0 = -mn_0$ . Then we have*

$$\varphi\left(t \frac{d}{dt}\right) \frac{1}{\det(I - tZ)^{n_0}} = \frac{1}{\det(I - tZ)^{n_0}} \sum_{k=0}^n \frac{D^k \varphi(x_0)}{k!} \sum_{|\lambda|=k} \frac{|\lambda|!}{z_\lambda} n_0^{\ell(\lambda)} p_\lambda\left(\frac{1}{I - tZ}\right), \quad (2.31)$$

where

$$\begin{aligned}
\lambda &= (1^{m_1(\lambda)} 2^{m_2(\lambda)} \dots) \quad (m_i(\lambda) \geq 0), \\
|\lambda| &:= \sum_i i m_i(\lambda), \quad \ell(\lambda) := \sum_i m_i(\lambda), \quad z_\lambda := \prod_i i^{m_i(\lambda)} m_i(\lambda)!, \\
p_\lambda(Z) &:= \prod_i (\text{Tr} Z^i)^{m_i(\lambda)}, \quad D^k \varphi(x_0) = \sum_{j=0}^k \binom{k}{j} (-1)^j \varphi(x_0 - j).
\end{aligned}$$

Combing Theorem 2.3 and Lemma 2.4, we obtain the explicit expression of the Bergman kernel  $K_\alpha$  of the Hilbert space  $\mathcal{H}_\alpha$  as follows.

**Theorem 2.5.** *Assume that*

$$\tilde{\chi}(x) := \chi_1(\mu x - p) \equiv \prod_{j=1}^r \left( \mu x - p + 1 + (j-1) \frac{a}{2} \right)_{1+b+(r-j)a}. \quad (2.32)$$

Let  $D^k \tilde{\chi}(x)$  be the  $k$ -order difference of  $\tilde{\chi}$  at  $x$ , that is

$$D^k \tilde{\chi}(x) = \sum_{j=0}^k \binom{k}{j} (-1)^j \tilde{\chi}(x-j). \quad (2.33)$$

Then (2.27) can be rewritten as

$$\begin{aligned} & K_\alpha(z, w; \bar{z}, \bar{w}) \\ &= \frac{\pi^{d+d_0}}{\mu^d C_{\Omega} \chi_1(0) \chi_2(0) V(\Omega) V(B^{d_0})} \left( \frac{1}{N(z, \bar{z})} \right)^{\mu \alpha} \sum_{k=0}^d \frac{D^k \tilde{\chi}(d)}{k!} \frac{(\alpha - d - d_0)_{k+d_0}}{\left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right)^{\alpha-d+k}}. \end{aligned} \quad (2.34)$$

*Proof.* Let  $x_0 = d - \alpha$ . By

$$\chi_1(\mu(\alpha + x) - p)|_{x=x_0-j} = \chi_1(\mu(d-j) - p) = \tilde{\chi}(d-j),$$

we have

$$D^k(\chi_1(\mu(\alpha + x) - p))|_{x=x_0} = D^k \tilde{\chi}(d).$$

Using (2.31) and

$$(x)_k = \sum_{|\lambda|=k} \frac{|\lambda|!}{z_\lambda} x^{\ell(\lambda)},$$

we have

$$\chi_1(\mu(\alpha + t \frac{d}{dt}) - p) \frac{1}{(1-tz)^{\alpha-d}} = \frac{1}{(1-tz)^{\alpha-d}} \sum_{k=0}^d \frac{D^k \tilde{\chi}(d)}{k!} \frac{(\alpha-d)_k}{(1-tz)^k}. \quad (2.35)$$

For the Cartan domain  $B^{d_0}$ , its Hua polynomial

$$\chi_2(x) = (x+1)_{d_0}.$$

Then

$$\chi_2(\alpha - d - d_0 - 1)(\alpha - d)_k = (\alpha - d - d_0)_{d_0+k}. \quad (2.36)$$

From (2.35) and (2.36), we get (2.34).  $\square$

### 3 The Rawnsley's $\varepsilon$ -functions for $\Omega^{B^{d_0}}(\mu)$ with the canonical metric $g(\mu)$

In this section we give the explicit expression of the Rawnsley's  $\varepsilon$ -function and the coefficients  $a_1, a_2$  of its expansion for the Cartan-Hartogs domain  $(\Omega^{B^{d_0}}(\mu), g(\mu))$ .

**Theorem 3.1.** *Let  $\alpha > \max\{d+d_0, \frac{p-1}{\mu}\}$ . Then the Rawnsley's  $\varepsilon$ -function associated to  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  can be written as*

$$\varepsilon_\alpha(z, w) = \frac{1}{\mu^d} \sum_{k=0}^d \frac{D^k \tilde{\chi}(d)}{k!} \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right)^{d-k} (\alpha - d - d_0)_{k+d_0} \quad (3.1)$$

(see (2.32) and (2.33) for the definition of the functions  $\tilde{\chi}(x)$  and  $D^k \tilde{\chi}(x)$  respectively).

*Proof.* By (2.34), we have

$$\begin{aligned} & \exp\{-\alpha\Phi(z, w)\}K_\alpha(z, w; \bar{z}, \bar{w}) \\ &= \frac{\pi^{d+d_0}}{\mu^d C_\Omega \chi_1(0) \chi_2(0) V(\Omega) V(B^{d_0})} \sum_{k=0}^d \frac{D^k \tilde{\chi}(d)}{k!} \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right)^{d-k} (\alpha - d - d_0)_{k+d_0}. \end{aligned}$$

From [12] and [13], we have

$$V(\Omega) = \frac{\pi^d}{C_\Omega \chi_1(0)}.$$

Since  $C_{B^{d_0}} = 1$ , it follows that

$$V(B^{d_0}) = \frac{\pi^{d_0}}{C_{B^{d_0}} \chi_2(0)} = \frac{\pi^{d_0}}{\chi_2(0)}.$$

Therefore, we obtain (3.1).  $\square$

**Corollary 3.2.** *The coefficients  $a_1$  and  $a_2$  of the expansion of the Rawnsley's  $\varepsilon$ -function  $\varepsilon_\alpha$ , that is, the coefficients of  $\alpha^{d+d_0-1}$  and  $\alpha^{d+d_0-2}$  in (3.1) respectively, are given by*

$$a_1(z, w) = \frac{1}{\mu^d} \frac{D^{d-1} \tilde{\chi}(d)}{(d-1)!} \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right) - \frac{(d+d_0)(d+d_0+1)}{2}, \quad (3.2)$$

$$\begin{aligned} a_2(z, w) &= \frac{1}{\mu^d} \frac{D^{d-2} \tilde{\chi}(d)}{(d-2)!} \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right)^2 - \\ &\quad \frac{1}{\mu^d} \frac{D^{d-1} \tilde{\chi}(d)}{(d-1)!} \left(\frac{(d+d_0)(d+d_0+1)}{2} - 1\right) \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right) + \\ &\quad \frac{1}{24} (d+d_0-1)(d+d_0)(d+d_0+1)(3(d+d_0)+2) \quad (d \geq 2), \end{aligned} \quad (3.3)$$

$$\begin{aligned} a_2(z, w) &= -\frac{\mu-1}{\mu} \left(\frac{(1+d_0)(2+d_0)}{2} - 1\right) \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right) + \\ &\quad \frac{1}{24} d_0(d_0+1)(d_0+2)(3d_0+5) \quad (d=1). \end{aligned} \quad (3.4)$$

*Proof.* Let

$$(\alpha - d - d_0)_{d_0+k} = \sum_{j=0}^{d_0+k} c_{d_0+k,j} \alpha^j. \quad (3.5)$$

Substituting (3.5) into (3.1), we obtain

$$\varepsilon_\alpha(z, w) = \sum_{j=0}^{d+d_0} \alpha^j \sum_{k=\max(j-d_0, 0)}^d \frac{c_{d_0+k,j}}{\mu^d} \frac{D^k \tilde{\chi}(d)}{k!} \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right)^{d-k}, \quad (3.6)$$

which implies

$$a_j(z, w) = \sum_{k=\max(d-j, 0)}^d \frac{c_{d_0+k, d+d_0-j}}{\mu^d} \frac{D^k \tilde{\chi}(d)}{k!} \left(1 - \frac{\|w\|^2}{N(z, \bar{z})^\mu}\right)^{d-k}. \quad (3.7)$$

By

$$\begin{aligned}(\alpha - d - d_0)_{d+d_0} &= \prod_{k=1}^{d+d_0} (\alpha - k), \\(\alpha - d - d_0)_{d+d_0-1} &= \prod_{k=2}^{d+d_0} (\alpha - k), \\(\alpha - d - d_0)_{d+d_0-2} &= \prod_{k=3}^{d+d_0} (\alpha - k),\end{aligned}$$

we have

$$c_{d+d_0-1, d+d_0-1} = c_{d+d_0-2, d+d_0-2} = 1, \quad (3.8)$$

$$c_{d+d_0, d+d_0-1} = - \sum_{k=1}^{d+d_0} k = - \frac{(d+d_0)(d+d_0+1)}{2}, \quad (3.9)$$

$$c_{d+d_0-1, d+d_0-2} = - \sum_{k=2}^{d+d_0} k = - \frac{(d+d_0)(d+d_0+1)}{2} + 1, \quad (3.10)$$

$$\begin{aligned}c_{d+d_0, d+d_0-2} &= \sum_{1 \leq i < j \leq d+d_0} ij = \frac{1}{2} \left\{ \left( \sum_{k=1}^{d+d_0} k \right)^2 - \sum_{k=1}^{d+d_0} k^2 \right\} \\&= \frac{1}{24} (d+d_0-1)(d+d_0)(d+d_0+1)(3(d+d_0)+2).\end{aligned} \quad (3.11)$$

For  $d > 1$ , substituting (3.8), (3.9), (3.10) and (3.11) into (3.7), we obtain (3.2) and (3.3).

For  $d = 1$ , we have

$$\tilde{\chi}(x) = \mu x - 1. \quad (3.12)$$

Thus, by (3.1), we get (3.4).  $\square$

In order to calculate  $D^{d-1}\tilde{\chi}$  and  $D^{d-2}\tilde{\chi}$ , we need Lemma 3.3 and 3.4 below.

**Lemma 3.3.** *Let  $\tilde{\chi}(x) := \prod_{j=1}^r (\mu x - p + 1 + (j-1)\frac{a}{2})_{1+b+(r-j)a} = \sum_{j=0}^d c_j x^{d-j}$ . Then*

$$c_0 = \mu^d, \quad (3.13)$$

$$c_1 = -\frac{1}{2}\mu^{d-1}dp, \quad (3.14)$$

$$\begin{aligned}c_2 &= \frac{1}{2}\mu^{d-2} \left\{ \frac{d^2 p^2}{4} - \frac{r(p-1)p(2p-1)}{6} + \frac{r(r-1)a(3p^2-3p+1)}{12} - \right. \\&\quad \left. \frac{(r-1)r(2r-1)a^2(p-1)}{24} + \frac{r^2(r-1)^2 a^3}{48} \right\}.\end{aligned} \quad (3.15)$$

*Proof.* The coefficients of  $x^d$ ,  $x^{d-1}$  and  $x^{d-2}$  in the function  $\tilde{\chi}(x)$  (see (2.32)), respectively, are given by

$$\begin{aligned}c_0 &= \mu^d, \\c_1 &= -\mu^d \sum_{j=1}^r \sum_{i=1}^{1+b+(r-j)a} \frac{1}{\mu} \left( p - i - (j-1)\frac{a}{2} \right),\end{aligned} \quad (3.16)$$

$$c_2 = \frac{1}{2}\mu^d \left\{ \left( -\frac{c_1}{\mu^d} \right)^2 - \sum_{j=1}^r \sum_{i=1}^{1+b+(r-j)a} \left( \frac{1}{\mu} \left( p - i - (j-1)\frac{a}{2} \right) \right)^2 \right\}. \quad (3.17)$$

Using (2.1) and formulas

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4},$$

we have

$$\begin{aligned} \sum_{j=1}^r \sum_{i=1}^{1+b+(r-j)a} \left( p - i - (j-1)\frac{a}{2} \right) &= \sum_{j=1}^r \sum_{k=1+(j-1)\frac{a}{2}}^{p-1-(j-1)\frac{a}{2}} k \\ &= \sum_{j=1}^r \frac{1}{2} p (p-1-(j-1)a) \\ &= \frac{1}{2} p \sum_{j=1}^r (b+1+(r-j)a) \\ &= \frac{1}{2} p \left( (1+b)r + \frac{r(r-1)}{2} a \right) \\ &= \frac{1}{2} p d, \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\sum_{j=1}^r \sum_{i=1}^{1+b+(r-j)a} \left( p - i - (j-1)\frac{a}{2} \right)^2 \\ &= \sum_{j=1}^r \sum_{k=1+(j-1)\frac{a}{2}}^{p-1-(j-1)\frac{a}{2}} k^2 \\ &= \frac{1}{24} \sum_{j=0}^{r-1} \{ (2p-2-ja)(2p-ja)(2p-1-ja) - ja(2+ja)(1+ja) \} \\ &= \frac{1}{24} \sum_{j=0}^{r-1} \{ (2p-2)(2p-1)2p - (12p^2-12p+4)aj + 6(p-1)a^2j^2 - 2a^3j^3 \} \\ &= \frac{r(p-1)p(2p-1)}{6} - \frac{r(r-1)a(3p^2-3p+1)}{12} + \\ &\quad \frac{(r-1)r(2r-1)a^2(p-1)}{24} - \frac{r^2(r-1)^2a^3}{48}. \end{aligned} \tag{3.19}$$

Combining (3.16), (3.17), (3.18) and (3.19), we get (3.14) and (3.15).  $\square$

**Lemma 3.4.** *For any polynomial  $f(x)$  in real variable  $x$ , take  $Df(x) := f(x) - f(x-1)$ . Let  $A_d = D^{d-1}x^d$ ,  $B_d = D^{d-2}x^d$ . Then we have*

$$A_d = \frac{d!}{2} (2x-d+1) \quad (d \geq 1), \tag{3.20}$$

$$B_d = \frac{d!}{24} \{ 12x^2 - 12(d-2)x + 3d^2 - 11d + 10 \} \quad (d \geq 2). \tag{3.21}$$

*Proof.* Firstly, we have the recurrence relations

$$\begin{aligned}
A_d &= D^{d-2}(Dx^d) \\
&= D^{d-2} \left( \sum_{j=0}^{d-1} (-1)^{d+1-j} \binom{d}{j} x^j \right) \\
&= dD^{d-2}x^{d-1} - \frac{d(d-1)}{2} D^{d-2}x^{d-2} \\
&= dA_{d-1} - \frac{d!}{2}
\end{aligned} \tag{3.22}$$

for  $d > 1$  and

$$\begin{aligned}
B_d &= D^{d-3}(Dx^d) \\
&= D^{d-3} \left( \sum_{j=0}^{d-1} (-1)^{d+1-j} \binom{d}{j} x^j \right) \\
&= dD^{d-3}x^{d-1} - \frac{d(d-1)}{2} D^{d-3}x^{d-2} + \frac{d(d-1)(d-2)}{6} D^{d-3}x^{d-3} \\
&= dB_{d-1} - \frac{d(d-1)}{2} A_{d-2} + \frac{d!}{6}
\end{aligned} \tag{3.23}$$

for  $d > 2$ . Now, by solving difference equation

$$\begin{cases} A_d = dA_{d-1} - \frac{d!}{2}, \\ B_d = dB_{d-1} - \frac{d(d-1)}{2} A_{d-2} + \frac{d!}{6}, \\ A_1 = x, \\ B_2 = x^2, \end{cases} \tag{3.24}$$

we obtain (3.20) and (3.21).  $\square$

Lemma 3.3 and Lemma 3.4 imply the following results.

**Lemma 3.5.** Suppose that  $D^{d-1}\tilde{\chi}(d)$  and  $D^{d-2}\tilde{\chi}(d)$  are defined by (2.32) and (2.33). Then we have

$$\frac{D^{d-1}\tilde{\chi}(d)}{(d-1)!} = \frac{d\mu^{d-1}}{2}(\mu(d+1) - p) \quad (d \geq 1), \tag{3.25}$$

$$\frac{D^{d-2}\tilde{\chi}(d)}{(d-2)!} = \mu^{d-2} \left\{ \frac{1}{24}(d-1)d(d+1)(3d+10)\mu^2 - \frac{1}{4}p(d-1)d(d+2)\mu + \frac{1}{2}\tilde{c}_2 \right\} \quad (d \geq 2), \tag{3.26}$$

where

$$\begin{aligned}
\tilde{c}_2 &= \frac{d^2p^2}{4} - \frac{r(p-1)p(2p-1)}{6} + \frac{r(r-1)a(3p^2-3p+1)}{12} - \\
&\quad \frac{(r-1)r(2r-1)a^2(p-1)}{24} + \frac{r^2(r-1)^2a^3}{48}.
\end{aligned} \tag{3.27}$$

*Proof.* From  $\tilde{\chi}(x) = c_0x^d + c_1x^{d-1} + c_2x^{d-2} + \dots + c_n$ , we get

$$\begin{cases} D^{d-1}\tilde{\chi}(x) = c_0A_d(x) + c_1(d-1)!, \\ D^{d-2}\tilde{\chi}(x) = c_0B_d(x) + c_1A_{d-1}(x) + c_2(d-2)!. \end{cases} \tag{3.28}$$

Let  $x = d$ , substituting (3.13), (3.14), (3.15), (3.20) and (3.21) into (3.28), we obtain (3.25) and (3.26).  $\square$



## 4 The proof of Theorem 1.3

*The proof of Theorem 1.3.* If the dimension of  $\Omega$  is 1 (i.e.,  $d = 1$ ), from (3.4), we get that  $a_2$  is constant if and only if  $\mu = 1$ , that is,  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is biholomorphically isometric to the complex hyperbolic space  $(B^{1+d_0}, g_{hyp})$ .

If the dimension of  $\Omega$  is larger than 1 (i.e.,  $d > 1$ ), it follows from (3.3) that the coefficient  $a_2$  of the expansion of the function  $\varepsilon_\alpha$  associated to  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is constant if and only if

$$\frac{D^{d-1}\tilde{\chi}(d)}{(d-1)!} = \frac{D^{d-2}\tilde{\chi}(d)}{(d-2)!} = 0. \quad (4.1)$$

From (3.25), (3.26) and (3.27), we get that  $a_2$  is constant if and only if

$$\mu = \frac{p}{d+1}, \quad (4.2)$$

and

$$12(d+1) \left\{ \frac{d^2 p^2}{4} - \frac{r(p-1)p(2p-1)}{6} + \frac{r(r-1)a(3p^2-3p+1)}{12} - \frac{(r-1)r(2r-1)a^2(p-1)}{24} + \frac{r^2(r-1)^2 a^3}{48} \right\} - (d-1)d(3d+2)p^2 = 0. \quad (4.3)$$

(1) For the bounded symmetric domain  $\Omega_I(m, n)$  ( $1 \leq m \leq n$ ), its rank  $r = m$ , the characteristic multiplicities  $a = 2, b = n - m$ , the dimension  $d = mn$ , the genus  $p = m + n$ . By (4.3), we obtain

$$2mn(m^2 - 1)(n^2 - 1) = 0, \quad (4.4)$$

that is,  $r = m = 1$ .

(2) For the bounded symmetric domain  $\Omega_{II}(2n)$  ( $n \geq 2$ ), its rank  $r = n$ , the characteristic multiplicities  $a = 4, b = 0$ , the dimension  $d = n(2n - 1)$ , the genus  $p = 2(2n - 1)$ . By (4.3), we obtain

$$4n^2(8n^4 - 20n^3 + 10n^2 + 5n - 3) = 0, \quad (4.5)$$

which is not satisfied by any positive integer  $n$  with  $n \geq 2$ .

(3) For the bounded symmetric domain  $\Omega_{III}(2n + 1)$  ( $n \geq 2$ ), its rank  $r = n$ , the characteristic multiplicities  $a = 4, b = 2$ , the dimension  $d = n(2n + 1)$ , the genus  $p = 4n$ . By (4.3), we obtain

$$4n(2n - 1)(2n + 1)^2(n - 1)(n + 1) = 0. \quad (4.6)$$

This equation has no positive integer solution  $n$  with  $n \geq 2$ .

(4) For the bounded symmetric domain  $\Omega_{III}(n)$  ( $n \geq 2$ ), its rank  $r = n$ , the characteristic multiplicities  $a = 1, b = 0$ , the dimension  $d = n(n + 1)/2$ , the genus  $p = n + 1$ . By (4.3), we obtain

$$\frac{1}{8}n^2(n^4 + 5n^3 + 5n^2 - 5n - 6) = 0. \quad (4.7)$$

The equation has no positive integer solution  $n$  with  $n \geq 2$ .

(5) For the bounded symmetric domain  $\Omega_{IV}(n)$  ( $n \geq 5$ ), its rank  $r = 2$ , the characteristic multiplicities  $a = n - 2, b = 0$ , the dimension  $d = n$ , the genus  $p = n$ . By (4.3), we obtain

$$n(n^2 + n - 2) = 0 \quad (4.8)$$

for  $n \geq 5$ , which is impossible.

(6) For the bounded symmetric domain  $\Omega_V(16)$ , its rank  $r = 2$ , the characteristic multiplicities  $a = 6, b = 4$ , the dimension  $d = 16$ , the genus  $p = 12$ . It means that (4.3) does not hold in this case.

(7) For the bounded symmetric domain  $\Omega_{VI}(27)$ , its rank  $r = 3$ , the characteristic multiplicities  $a = 8, b = 0$ , the dimension  $d = 27$ , the genus  $p = 18$ . It is obvious that (4.3) does not hold in this case.

Combing the above results, we get that  $a_2$  is constant if and only if the rank  $r = 1$  of the bounded symmetric domain  $\Omega$  and  $\mu = \frac{p}{d+1} = 1$ , which implies that the Cartan-Hartogs domain  $(\Omega^{B^{d_0}}(\mu), g(\mu))$  is biholomorphically isometric to the complex hyperbolic space  $(B^{d+d_0}, g_{hyp})$ .  $\square$

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